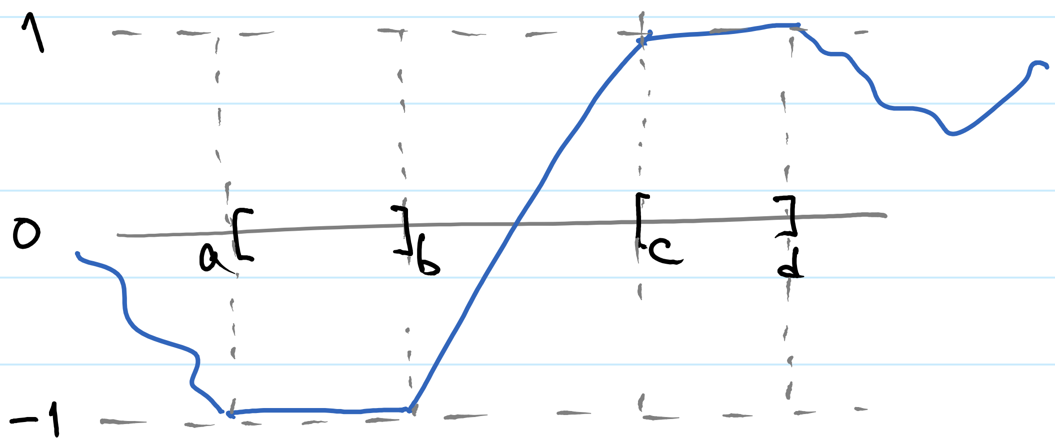
Feb 14, 2017

Tuesday, February 14, 2017 10:06 AM

Let us start with an easy question.

Given $[a,b], [c,d], b<c$

Find a continuous $f: \mathbb{R} \to [-1,1]$ such that $f|_{[a,b]} \equiv -1$, $f|_{[c,d]} \equiv 1$



The crucial task is to define f on $[b,c]$.

$$f(x) = -1 + \frac{2}{c-b}(x-b) = \frac{2x-b-c}{c-b}$$

Proposition. Let (X,d) be a metric space; $A, B \subset X$ be closed and $A \cap B = \phi$. Then $\exists$ continuous $f: X \longrightarrow [-1,1]$ such that $f|_A \equiv -1$ and $f|_B \equiv 1$.

The method above for $X = \mathbb{R}$ is insightful

$$f(x) = \frac{2x-b-c}{c-b} = \frac{(x-b)-(c-x)}{(x-b)+(c-x)}$$

Idea of Proof. Define $f : X \longrightarrow [-1,1]$ by

$$f(x) = \frac{d(x,A) - d(x,B)}{d(x,A) + d(x,B)}$$

Clearly, $f(x) = -1$ when $x \in A$, and
$f(x) = 1$ when $x \in B$

For continuity of f, there are several essential steps.

(i) $x \longmapsto d(x,S)$ is continuous

This follows from $x \longmapsto d(x, y_0)$ is continuous and $d(x,S)$ is the infinmum

(ii) The denominator $d(x,A) + d(x,B) \neq 0$

It is zero $\Leftrightarrow$ both $d(x,A) = 0 = d(x,B)$

Moreover, $d(x,S) = 0 \Rightarrow x \in \overline{S}$

Thus $d(x,A) = 0 = d(x,B)$ only occur if $x \in \overline{A} = A$ and $x \in \overline{B} = B$.

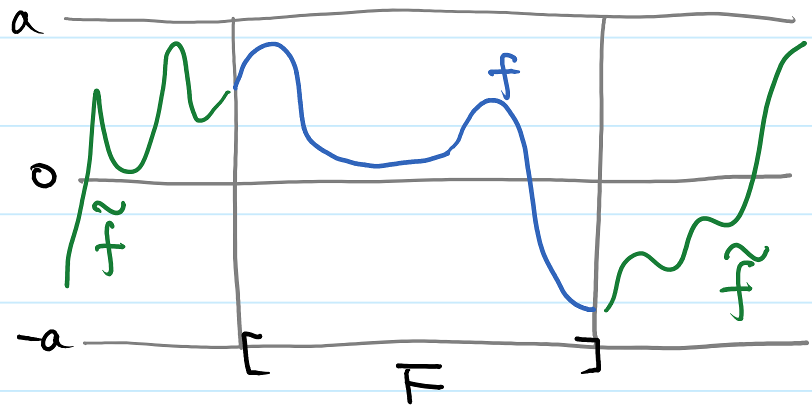
This won't occur as $A \cap B = \emptyset$

Urysohn Lemma. If a space is normal then the same conclusion of the proposition holds.

The definition of normal spaces will be discussed later.

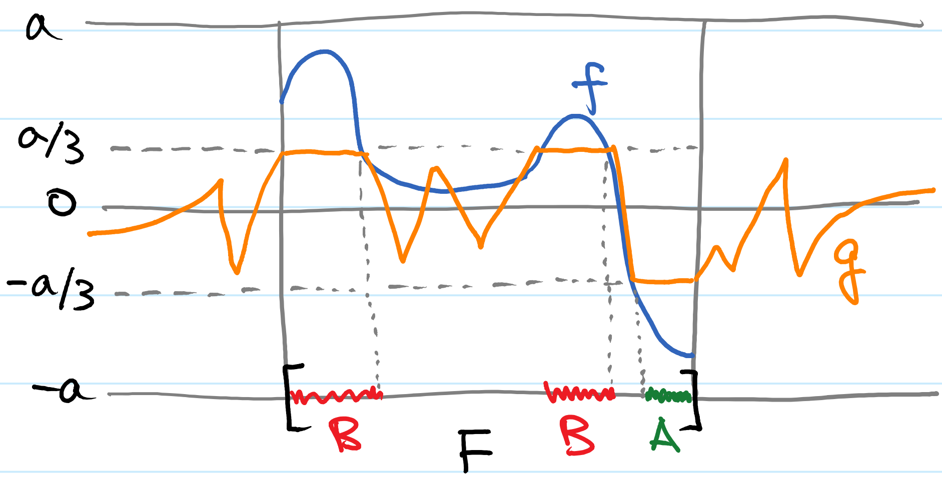
Tietz Extension Theorem Let X be a space satisfying the proposition (e.g. normal, metric). If $F \subset X$ is closed and $f: F \longrightarrow [-a,a] \subset \mathbb{R}$ is continuous then $\exists$ continuous extension $\tilde{f}: X \to [-a,a]$, $\tilde{f}|_F \equiv f$.

Illustration



We will construct $\tilde{f}$ step by step.

First step:



Let $A = f^{-1}[-a, \frac{-a}{3}]$ and $B = f^{-1}[\frac{a}{3}, a]$

By Proposition–Property, $\exists$ continuous
$g: X \longrightarrow [\frac{-a}{3}, \frac{a}{3}]$ where

$g|_A \equiv \frac{-a}{3}$ and $g|_B = \frac{a}{3}$.

From now on, we call this g_1 from step one.

Observe that

① On X, $\|g_1\| = \sup\{|g_1(x)| : x \in X\} \leq \frac{a}{3}$

② On F, $\|f - g_1\| = \sup\{|f(x) - g_1(x)| : x \in F\} \leq \frac{2a}{3}$

Next step, repeat the work on

$$f - g_1 : F \longrightarrow \left[\frac{-2a}{3}, \frac{2a}{3}\right]$$

There is a continuous

$$g_2 : X \longrightarrow \left[\frac{-2a}{9}, \frac{2a}{9}\right] \quad \text{and}$$

$$(f - g_1) - g_2 : F \longrightarrow \left[\frac{-4a}{9}, \frac{4a}{9}\right] \quad \text{and so on.}$$

That is,

$$g_n : X \longrightarrow \left[\frac{-a}{3}\left(\frac{2}{3}\right)^{n-1}, \frac{a}{3}\left(\frac{2}{3}\right)^{n-1}\right]$$

$$f - \sum_{k=1}^{n} g_k : F \longrightarrow \left[-\left(\frac{2}{3}\right)^n a, \left(\frac{2}{3}\right)^n a\right]$$

By standard theory in analysis,

$$\sum_{k=1}^{n} g_k \xrightarrow[\text{uniformly}]{} \tilde{f} : X \longrightarrow [-a, a]$$

Moreover,

$$\tilde{f}|_F \equiv f \quad \text{as} \quad \|f - \tilde{f}\|_F \leq \varepsilon \quad \forall \varepsilon > 0.$$

For the content of complete metric spaces, please refer to MATH3060 or additional notes. It is only a topic on metric spaces.

First, Cauchy sequence is defined. Then in Complete metric space, every Cauchy sequence converges. A major result is the Contraction Mapping Theorem. It is useful to prove several important results, e.g., Newton's Method, Inverse/Implicit Function Theorem, and Existence of solution to ODE.

To refresh memory, let us prove

Proposition. In a complete metric space X, $Y \subset X$ is complete $\Leftrightarrow$ Y is closed in X.

"$\Rightarrow$" Let $(y_n)_{n=1}^{\infty}$ be a Cauchy sequence in Y. It is a Cauchy sequence in X and so $\exists$ $x \in X$ such that $y_n \to x$.

Since $y_n \in Y \to x$, $x \in \overline{Y} = Y$

"$\Leftarrow$" Let $x \in \overline{Y}$, As X is metric, $\exists$ $(y_n)_{n=1}^{\infty}$ in Y $y_n \to x$; and so it is Cauchy. Since Y is complete, $y_n \to y \in Y$. By uniqueness $x = y \in Y$.

02-14-p6

Monday, February 20, 2017 11:30 PM

Recall that one may prove Intermediate Value Theorem by an argument of Nested Intervals. On second thought, the "intermediate value" only occurs because the space $\mathbb{R}$ "has no hole"; i.e., complete! Here is an analogue.

Cantor Intersection Theorem. Let X be a complete metic space. If

* each $F_n \subset X$ is closed
* $F_{n+1} \subset F_n$ for all n
* $\operatorname{diam}(F_n) \to 0$ as $n \to \infty$

then $\bigcap_{n=1}^{\infty} F_n$ is a singleton — existence, uniqueness

Idea of Proof.

For each F_n, pick $x_n \in F_n$

Use the assumption $\operatorname{diam}(F_n) \to 0$ to show that $(x_n)_{n=1}^{\infty}$ is Cauchy. Thus by the completeness of X, $x_n \longrightarrow x \in X$.

Note that for all N, $(x_n)_{n=N}^{\infty}$ is in F_N and $x_n \to x$, $\therefore x \in \overline{F_N} = F_N$

Thus, $x \in \bigcap_{n=1}^{\infty} F_n$

Finally, by $\operatorname{diam}(F_n) \to 0$, $\bigcap_{n=1}^{\infty} F_n = \{x\}$.